Pergamon

Int. J. Solids Structures Vol. 33, No. 19, pp. 2695 2718, 1996

0020-7683/96 $15.00 + .00

0020-7683(95)00183-2

THE DUAL BOUNDARY ELEMENT METHOD FOR TRANSIENT THERMOELASTIC CRACK PROBLEMS

N. N. V. PRASAD, M. H. ALIABADI
Wessex Institute of Technology, Ashurst Lodge, Ashurst, Southampton SO40 7AA, U.K.

and

D. P. ROOKE
Structural Materials Centre, DRA, Farnborough GU14 6TD, U.K.

(*Received* 1 *May* 1995; *in revised form* 20 *July* 1995)

Abstract—A two-dimensional boundary element formulation, which does not require domain discretization and allows a single-region analysis, is presented for transient thermoelastic crack problems. The formulation is based on the dual boundary element method; temperature and displacement equations are used on one crack surface, flux and traction equations on the other. Time is modelled with linear time elements. Stress intensity factors are calculated using the path independent $\hat{J}$-integral. Several crack problems are solved and the results are compared, where possible, with existing solutions. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Under certain conditions thermal and mechanical loads can lead to the fracture of engineering components. In sensitive equipment such as pressure vessels, the fracture of a component, due to sudden cooling, say, can lead to complete failure. The possibility of a crack-induced failure following thermal shock can be assessed by calculating the thermal stress intensity factors for the cracked component. This paper presents a numerical technique for the accurate calculation of stress intensity factors as a function of time for uncoupled transient thermoelastic problems.

Analytical results for transient thermal problems exist for only a few cracked configurations. Numerical methods like the finite element method (FEM) and the boundary element method (BEM) are the most popular techniques used for the analysis of transient problems. Many papers have been published on the use of the finite element method (e.g. Emmel and Stamm, 1985; Emery *et al.*, 1977; Hellen *et al.*, 1982). However, the boundary element method offers many advantages, for example boundary only discretization and simple modelling for crack growth studies.

Uncoupled transient thermoelasticity has been the subject of many investigations with a boundary element method of analysis. For instance, Tanaka *et al.* (1984a) implemented a volume based thermal body approach. However volume discretization removes some of the advantages of the standard BEM. Sládek and Sládek (1989, 1992) presented a series of papers on coupled thermoelasticity which included a time-domain method. The initial time-domain boundary integral equations were presented in a boundary only formulation, but the primary variables included time derivatives. Sládek and Sládek (1989) later presented a boundary integral formulation in terms of regular primary variables; they used inverse Laplace transforms on their previous equations. Their work was later summarized in Sládek and Sládek (1992). The same equations were obtained by Dargush and Banerjee (1989) who used the reciprocal theorem of Ionescu-Cazimir (1964). Sládek *et al.* (1989) presented fundamental solutions of displacement and traction only and their time integrations.

The literature on applications of thermoelasticity to crack problems is limited (Tanaka *et al.*, (1984b), Sládek *et al.* (1992), Raveendra and Banerjee (1992)). All the early solutions of crack problems were restricted to symmetric problems, so as to avoid collocating on both crack surfaces which leads to singular matrices. Tanaka *et al.* (1984b) applied a subregions technique, developed by Blandford *et al.* (1981) for fracture problems which involved domain discretization. Raveendra *et al.* (1992) also used a subregion technique to solve crack problems using a boundary only formulation.

Portela *et al.* (1992) presented a formulation called the dual boundary element method (DBEM) which facilitates the analysis of arbitrary crack problems in a single region. They overcame the problem of singularities in the final system of equations by using different equations on the two crack surfaces. They collocated using a displacement equation on one crack face and a traction equation on the opposite face. This method was later extended to steady-state thermoelasticity by Prasad *et al.* (1994a).

In this paper a boundary formulation of uncoupled transient thermoelasticity is implemented for crack problems, which makes use of the dual boundary element method. The advantage of the boundary formulation is that it removes the necessity of any domain discretization. The advantage in using DBEM is that the domain does not have to be sectioned along an artificial boundary. Thus, DBEM simplifies the modelling of crack propagation also. In the current formulation, temperature and displacement boundary integral equations are applied on one crack surface, and flux and traction equations are applied on the opposite surface. Boundary and internal formulations for all the four equations are presented. Time integration is done using constant and linear time steps. Accurate values of stress intensity factors are obtained from the $\hat{J}$-integral. Two example problems are presented and the results are compared to existing work where available.

DUAL BOUNDARY ELEMENT METHOD

Consider a linear, elastic, isotropic and homogeneous body occupying a domain Ω enclosed by a boundary Γ. The governing sets of equations for uncoupled transient thermoelasticity are the diffusion equation and the equations of elasticity. The equations can be expressed as follows:

$$\theta_{,jj} - \frac{1}{\kappa}\dot{\theta} = 0 \tag{1}$$

and

$$\mu u_{i,jj} + \frac{\mu}{(1-2\nu)} u_{j,ij} - \frac{2\mu(1+\nu)}{(1-2\nu)} \alpha\theta_{,i} = 0; \tag{2}$$

where θ is the temperature, u_i is the displacement component, μ is the shear modulus, ν is Poisson's ratio, α is the coefficient of linear expansion and κ is the diffusivity (see Appendix A). In the above equations, a subscript i preceded by a comma represents differentiation with respect to the ith spatial coordinate in the Cartesian system; repeated indices imply summation; and the dot over θ represents differentiation with respect to the time τ.

The differential eqn (1) is solved subject to initial temperature conditions in Ω and temperature and flux boundary conditions on Γ_θ and Γ_q respectively. The flux q is defined by $q = -\lambda\theta_{,n}$, where n preceded by a comma represents differentiation with respect to the outward normal. The differential eqn (2) requires displacement and traction boundary conditions on Γ_u and Γ_t respectively. The traction t_i is defined by $t_i = \sigma_{ij}n_j$, where σ_{ij} is the stress tensor and n_j is the normal vector.

For eqn (1):

$$\text{initial conditions}(\tau = \tau_o) \quad \theta(X, \tau) = \theta(X, \tau_o) \text{ where } X \in \Omega;$$

$$\text{boundary conditions} \quad \theta(x, \tau) = \bar{\theta}(x, \tau) \text{ where } x \in \Gamma_\theta$$

$$\text{and} \qquad q(x,\tau) = \bar{q}(x,\tau) \text{ where } x \in \Gamma_q. \tag{3}$$

For eqn (2):

$$\text{boundary conditions} \quad u_i(x) = \bar{u}_i(x) \text{ where } x \in \Gamma_u$$
$$\text{and} \qquad t_i(x) = \bar{t}_i(x) \text{ where } x \in \Gamma_t \tag{4}$$

where X represents an internal point and x represents a boundary point.

The standard temperature equation for an interior point X' at time τ_F can be generated from weighted residual statements, as shown in El-Zafrany (1993) and Brebbial *et al.* (1984); namely

$$\theta(X',\tau_F) - \int_\Gamma \int_{\tau_o}^{\tau_F} \theta(x,\tau) Q(X',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$= -\int_\Gamma \int_{\tau_o}^{\tau_F} q(x,\tau)\Theta(X',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x) + \int_\Omega \theta(X,\tau_o)\Theta(X',X,\tau_F,\tau_o)\,\mathrm{d}\Omega(X), \tag{5}$$

where the fundamental solution Θ is the solution of the following equation (Carslaw and Jaeger, 1959);

$$\lambda \nabla^2 \Theta(X',X,\tau_F,\tau) + c_\varepsilon \dot{\Theta}(X',X,\tau_F,\tau) = -\delta(X',X)\delta(\tau_F,\tau). \tag{6}$$

In the above equation $\delta(X',X)$ is the Dirac delta function with the following properties:

$$f(X') = \int_\Omega f(X)\delta(X',X)\,\mathrm{d}X. \tag{7}$$

In eqn (5), Q can be obtained from Θ by using the flux-temperature relationship. As the initial temperature $\theta(X,\tau_o)$ satisfies the steady state equation, then the domain integral in eqn (5) can be converted into boundary integrals (El-Zafrany, 1993). The resulting boundary integral equation is

$$\theta(X',\tau_F) - \theta(X',\tau_o) - \int_\Gamma \int_{\tau_o}^{\tau_F} \theta(x,\tau) Q(X',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$= -\int_\Gamma \int_{\tau_o}^{\tau_F} q(x,\tau)\Theta(X',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$-\int_\Gamma \{\theta(x,t_o)Q^o(X',x,\tau_F,\tau_o) - q(x,t_o)\Theta^o(X',x,\tau_F,\tau_o)\}\,\mathrm{d}\Gamma(x), \tag{8}$$

where the term $\theta(X',\tau_o)$ comes from the conversion of the domain integral to a boundary integral. The last two integrals also come from the conversion of the domain integral, due to the initial temperature, to boundary integrals; Θ^o and Q^o are given in Appendix A.

The displacement boundary integral equation for an internal point X' can be obtained either by use of Laplace transformations as Sládek and Sládek (1992), or by using the reciprocal theorem Dargush and Banerjee (1989). The result is

$$u_i(X',\tau_F)+\int_\Gamma T_{ij}(X',x)u_j(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}F_i(X',x,\tau_F,\tau)\theta(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$=\int_\Gamma U_{ij}(X',x)t_j(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}G_i(X',x,\tau_F,\tau)q(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x) \quad (9)$$

where U_{ij} is the displacement fundamental solution due to unit line load and G_i is the displacement fundamental solution due to a unit heat flow applied at τ_o; the functions T_{ij} and F_i are the traction and flux fundamental solutions; they can be obtained by applying the constitutive law on U_{ij} and G_i.

Temperature and flux equations

Temperature equations for a boundary point x' can be obtained by moving the internal point in eqn (8) to the boundary. The temperature equation for a point x' is

$$c(x')(\theta(x',\tau_F)-\theta(x',\tau_o))-\int_\Gamma\int_{\tau_o}^{\tau_F}\theta(x,\tau)Q(x',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$=-\int_\Gamma\int_{\tau_o}^{\tau_F}q(x,\tau)\Theta(x',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$-\int_\Gamma\{\theta(x,\tau_o)Q^o(x',x,\tau_F,\tau_o)-q(x,\tau_o)\Theta^o(x',x,\tau_F,\tau_o)\}\,\mathrm{d}\Gamma(x), \quad (10)$$

where $\int_\Gamma$ represents the Cauchy principal value integral; and $c(x')$ depends on the position of the point x' on the boundary. The function $c(x')$ multiplying the temperature at time τ_F comes from the spatial integrals of Θ and Q after a time integration. Similarly, the function $c(x')$ multiplying the temperature at time τ_o comes from the spatial integrals of Θ^o and Q^o. After the time integration of the Θ and Q integrals in eqn (10), the resulting terms have $O(\ln r)$ and $O(1/r)$ spatial singularities, respectively, the same as in the steady state temperature equation (where $r=|x'-x|$).

If the point x'_c is on one of the crack surfaces, and x''_c is the point on the opposite crack surface with the same coordinates as x'_c, then the temperature equation for x'_c is

$$c(x'_c)(\theta(x'_c,\tau_F)-\theta(x'_c,\tau_o))+c(x''_c)(\theta(x''_c,\tau_F)-\theta(x''_c,\tau_o))$$

$$-\int_\Gamma\int_{\tau_o}^{\tau_F}\theta(x,\tau)Q(x'_c,x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)=-\int_\Gamma\int_{\tau_o}^{\tau_F}q(x,\tau)\Theta(x'_c,x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$-\int_\Gamma\{\theta(x,\tau_o)Q^o(x'_c,x,\tau_F,\tau_o)-q(x,\tau_o)\Theta^o(x'_c,x,\tau_F,\tau_o)\}\,\mathrm{d}\Gamma(x) \quad (11)$$

If the crack surface is smooth at x'_c, then $c(x'_c)=c(x''_c)=(1/2)$.

The equation for temperature derivatives at an internal point X' can be obtained by differentiating eqn (8) with respect to X' to give the following equation:

$$\left(\frac{\partial\theta(X',\tau_F)}{\partial X'_i}-\frac{\partial\theta(X',\tau_o)}{\partial X'_i}\right)-\int_\Gamma\int_{\tau_o}^{\tau_F}\theta(x,\tau)\frac{\partial Q(X',x,\tau_F,\tau)}{\partial X'_i}\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$=-\int_\Gamma\int_{\tau_o}^{\tau_F}q(x,\tau)\frac{\partial\Theta(X',x,\tau_F,\tau)}{\partial X'_i}\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$-\int_\Gamma\left\{\theta(x,\tau_o)\frac{\partial Q^o(X',x,\tau_F,\tau_o)}{\partial X'_i}-q(x,\tau_o)\frac{\partial\Theta^o(X',x,\tau_F,\tau_o)}{\partial X'_i}\right\}\mathrm{d}\Gamma(x) \quad (12)$$

The singularity orders of the coefficients of θ and q in the integrals can be obtained after time integration. The orders of singularities can be seen to be the same as in the steady state flux equation, that is $O(1/r^2)$ and $O(1/r)$ for temperature and flux coefficients, respectively. When the collocation point is taken to a smooth boundary at x', the derivatives of the fundamental solutions can be expanded in a Taylor's series and the following relations obtained (Appendix C):

$$\lim_{X'\to x'}\int_\Gamma \left\{\int_{\tau_o}^{\tau_F} \frac{\partial Q(X',x,\tau_F,\tau)}{\partial X'_i}\theta(x,\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma(x)$$
$$= \frac{\theta_{,i}(x',\tau_F)}{4} + \oint\!\!\!\!=_\Gamma \left\{\int_{\tau_o}^{\tau_F} \frac{\partial Q(x',x,\tau_F,\tau)}{\partial x'_i}\theta(x,\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma(x) \quad (13)$$

$$\lim_{X'\to x'}\int_\Gamma \left\{\int_{\tau_o}^{\tau_F} \frac{\partial \Theta(X',x,\tau_F,\tau)}{\partial X'_i}q(x,\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma(x)$$
$$= -\frac{\theta_{,i}(x',\tau_F)}{4} + \fint_\Gamma \left\{\int_{\tau_o}^{\tau_F} \frac{\partial \Theta(x',x,\tau_F,\tau)}{\partial x'_i}q(x,\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma(x). \quad (14)$$

Notice that since $q = -\lambda\theta_{,k}n_k$ in eqn (14) and n_k depends on x, only $\theta_{,k}$ can be brought outside the integral during the calculation of the singular integral. In the above equations $\oint\!\!\!\!=$ represents the Hadamard principal value integral and $\fint$ represents the Cauchy principal value integral. Jump terms for the integrals of the initial conditions in eqn (12) can be calculated in a similar way. The temperature derivative equation for a point on a smooth boundary can now be written as follows:

$$\left(\frac{1}{2}\frac{\partial\theta(x',\tau_F)}{\partial x'_i} - \frac{1}{2}\frac{\partial\theta(x',\tau_o)}{\partial x'_i}\right) - \oint\!\!\!\!=_\Gamma \int_{\tau_o}^{\tau_F} \theta(x,\tau)\frac{\partial Q(x',x,\tau_F,\tau)}{\partial x'_i}\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$= -\fint_\Gamma \int_{\tau_o}^{\tau_F} q(x,\tau)\frac{\partial \Theta(x',x,\tau_F,\tau)}{\partial x'_i}\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$-\left[\oint\!\!\!\!=_\Gamma \theta(x,\tau_o)\frac{\partial Q^o(x',x,\tau_F,\tau_o)}{\partial x'_i}\mathrm{d}\Gamma(x) - \fint_\Gamma q(x,\tau_o)\frac{\partial \Theta^o(x',x,\tau_F,\tau_o)}{\partial x'_i}\mathrm{d}\Gamma(x)\right]. \quad (15)$$

Temperature derivatives for a point x'_c on a smooth crack surface are related to those for the equivalent point x''_c on the opposite crack surface as follows:

$$\left(\frac{1}{2}\frac{\partial\theta(x'_c,\tau_F)}{\partial x'_i} - \frac{1}{2}\frac{\partial\theta(x'_c,\tau_o)}{\partial x'_i}\right) + \left(\frac{1}{2}\frac{\partial\theta(x''_c,\tau_F)}{\partial x'_i} - \frac{1}{2}\frac{\partial\theta(x''_c,\tau_o)}{\partial x'_i}\right)$$
$$-\oint\!\!\!\!=_\Gamma \int_{\tau_o}^{\tau_F} \theta(x,\tau)\frac{\partial Q(x'_c,x,\tau_F,\tau)}{\partial x'_i}\mathrm{d}\tau\,\mathrm{d}\Gamma(x) = -\fint_\Gamma \int_{\tau_o}^{\tau_F} q(x,\tau)\frac{\partial \Theta(x'_c,x,\tau_F,\tau)}{\partial x'_i}\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$-\left[\oint\!\!\!\!=_\Gamma \theta(x,\tau_o)\frac{\partial Q^o(x'_c,x,\tau_F,\tau_o)}{\partial x'_i}\mathrm{d}\Gamma(x) - \fint_\Gamma q(x,\tau_o)\frac{\partial \Theta^o(x'_c,x,\tau_F,\tau_o)}{\partial x'_i}\mathrm{d}\Gamma(x)\right]. \quad (16)$$

The flux equation for the point x'_c can be obtained from the relationship between flux and temperature derivatives and is as follows:

$$\frac{1}{2}(q(x_c',\tau_F)-q(x_c',\tau_o))-\frac{1}{2}(q(x_c'',\tau_F)-q(x_c'',\tau_o))$$

$$+n_i(x_c')\;\text{⨎}_\Gamma\int_{\tau_o}^{\tau_F}Q_i(x_c',x,\tau_F,\tau)\theta(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$=n_i(x_c')\;\text{⨍}_\Gamma\int_{\tau_o}^{\tau_F}\Theta_i(x_c',x,\tau_F,\tau)q(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$

$$+n_i(x_c')\left[\;\text{⨎}_\Gamma Q_i^o(x_c',x,\tau_F,\tau_o)\theta(x,\tau_o)\,\mathrm{d}\Gamma(x)-\text{⨍}_\Gamma\Theta_i^o(x_c',x,\tau_F,\tau_o)q(x,\tau_o)\,\mathrm{d}\Gamma(x)\right],\tag{17}$$

where the condition $n_i(x_c') = -n_i(x_c'')$ is used and Q_i, Θ_i, Q_i^o and Θ_i^o are given in Appendix A.

Displacement and traction equations

The displacement equation for a collocation point x' can be obtained by taking the internal point X' to the boundary. Of the four integrands in the displacement eqn (9), U_{ij} and F_i, after the time integration, are weakly singular of $O(\ln r)$; G_i after time integration is not singular; and T_{ij} is strongly singular of $O(1/r)$. The singularity in T_{ij} gives a jump term at x'. Thus the displacement equation for the boundary point x' is given by

$$c_{ij}(x')u_j(x',\tau_F)+\text{⨍}_\Gamma T_{ij}(x',x)u_j(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}F_i(x',x,\tau_F,\tau)\theta(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$=\int_\Gamma U_{ij}(x',x)t_j(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}G_i(x',x,\tau_F,\tau)q(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x),\tag{18}$$

where c_{ij} depends on the position of x'. If x_c' is a point on a crack surface, and x_c'' is the corresponding point on the opposite crack surface, the displacement equation is

$$c_{ij}(x_c')u_j(x_c',\tau_F)+c_{ij}(x_c'')u_j(x_c'',\tau_F)$$
$$+\text{⨍}_\Gamma T_{ij}(x_c',x)u_j(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}F_i(x_c',x,\tau_F,\tau)\theta(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$=\int_\Gamma U_{ij}(x_c',x)t_j(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}G_i(x_c',x,\tau_F,\tau)q(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x),\tag{19}$$

where $c_{ij}(x_c') = c_{ij}(x_c'') = \frac{1}{2}\delta_{ij}$ for a smooth surface.

Traction equations are obtained by substituting the derivatives, with respect to X', of the displacement eqn (9), in the constitutive law. Derivatives of the displacement are given by the following:

$$\frac{\partial u_i(X',\tau_F)}{\partial X_j'}+\int_\Gamma\frac{\partial T_{ik}(X',x)}{\partial X_j'}u_k(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}\frac{\partial F_i(X',x,\tau_F,\tau)}{\partial X_j'}\theta(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x)$$
$$=\int_\Gamma\frac{\partial U_{ik}(X',x)}{\partial X_j'}t_k(x,\tau_F)\,\mathrm{d}\Gamma(x)+\int_\Gamma\int_{\tau_o}^{\tau_F}\frac{\partial G_i(X',x,\tau_F,\tau)}{\partial X_j'}q(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma(x).\tag{20}$$

By combining the above eqn (20) with the constitutive law in thermoelasticity, that is

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \frac{2\nu\mu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij} - \frac{2\mu(1+\nu)}{(1-2\nu)}\alpha\theta\delta_{ij}, \tag{21}$$

the stress equation for an interior point X' can be written as follows:

$$\begin{aligned}\sigma_{ij}(X',\tau_F) &+ \int_\Gamma T_{kij}(X',x)u_k(x,\tau_F)\,d\Gamma(x) \\ &+ \int_\Gamma\int_{\tau_o}^{\tau_F} F_{ij}(X',x,\tau_F,\tau)\theta(x,\tau)\,d\tau\,d\Gamma(x) + \frac{2\mu(1+\nu)}{(1-2\nu)}\alpha\theta(X',\tau_F)\delta_{ij} \\ &= \int_\Gamma U_{kij}(X',x)t_k(x,\tau_F)\,d\Gamma(x) + \int_\Gamma\int_{\tau_o}^{\tau_F} G_{ij}(X',x,\tau_F,\tau)q(x,\tau)\,d\tau\,d\Gamma(x),\end{aligned} \tag{22}$$

where T_{kij}, U_{kij}, F_{ij} and U_{ij} are given in Appendix A. In the stress equation T_{kij} is hypersingular of $O(1/r^2)$; U_{kij} and the time-integrated F_{ij} are strongly singular of $O(1/r)$; and G_{ij} after time integration is weakly singular of $O(\ln r)$. In elastostatics, it can be shown (see Cruse, 1977) that the summation of the singularities of the kernels of T_{kij} and U_{kij} gives $-\sigma(x')/2$, when the collocation point is taken to a smooth boundary. In transient thermoelasticity the singularities can be shown to be (Appendix C),

$$\begin{aligned}\lim_{X'\to x'} &\left\{\int_\Gamma T_{kij}(X',x)u_k(x,\tau_F)\,d\Gamma(x) - \int_\Gamma U_{kij}(X',x)t_k(x,\tau_F)\,d\Gamma(x)\right\} \\ &= \int\!\!\!\!\!\!=_\Gamma T_{kij}(x',x)u_k(x,\tau_F)\,d\Gamma(x) - \int\!\!\!\!\!\!-_\Gamma U_{kij}(x',x)t_k(x,\tau_F)\,d\Gamma(x) \\ &\quad - \frac{\sigma_{ij}(x',\tau_F)}{2} - \frac{\mu(1+\nu)\alpha}{(1-2\nu)}\theta(x',\tau_F)\delta_{ij} + \frac{\mu(1+\nu)\alpha}{2(1-\nu)(1-2\nu)}\theta(x',\tau_F)\delta_{ij}\end{aligned} \tag{23}$$

and

$$\begin{aligned}\lim_{X'\to x'} &\int_\Gamma \left\{\int_{\tau_o}^{\tau_F} F_{ij}(X',x,\tau_F,\tau)\theta(x,\tau)\,d\tau\right\}d\Gamma(x) \\ &= \int\!\!\!\!\!\!-_\Gamma \left\{\int_{\tau_o}^{\tau_F} F_{ij}(x',x,\tau_F,\tau)\theta(x,\tau)\,d\tau\right\}d\Gamma(x) - \frac{\mu(1+\nu)\alpha}{2(1-\nu)(1-2\nu)}\theta(x',\tau_F)\delta_{ij}.\end{aligned} \tag{24}$$

After substituting the above two equations in eqn (22), the following equation for the stress at a smooth boundary point x', is obtained

$$\begin{aligned}\frac{\sigma_{ij}(x',\tau_F)}{2} &+ \int\!\!\!\!\!\!=_\Gamma T_{kij}(x',x)u_k(x,\tau_F)\,d\Gamma(x) \\ &+ \int\!\!\!\!\!\!-_\Gamma\int_{\tau_o}^{\tau_F} F_{ij}(x',x,\tau_F,\tau)\theta(x,\tau)\,d\tau\,d\Gamma(x) + \frac{\mu(1+\nu)\alpha}{(1-2\nu)}\theta(x',\tau_F)\delta_{ij} \\ &= \int\!\!\!\!\!\!-_\Gamma U_{kij}(x',x)t_k(x,\tau_F)\,d\Gamma(x) + \int_\Gamma\int_{\tau_o}^{\tau_F} G_{ij}(x',x,\tau_F,\tau)q(x,\tau)\,d\tau\,d\Gamma(x).\end{aligned} \tag{25}$$

The stress equation for a point x_c' on a smooth crack surface and the corresponding point $x_{\bar{c}}'$ on the opposite crack surface is as follows:

$$\frac{\sigma_{ij}(x_c', \tau_F)}{2} + \frac{\sigma_{ij}(x_c'', \tau_F)}{2} + \displaystyle{=}\!\!\!\!\!\!\int_\Gamma T_{kij}(x_c', x) u_k(x, \tau_F)\, \mathrm{d}\Gamma(x)$$

$$+ \displaystyle{=}\!\!\!\!\!\!\int_\Gamma \int_{\tau_o}^{\tau_F} F_{ij}(x_c', x, \tau_F, \tau)\theta(x, \tau)\, \mathrm{d}\tau\, \mathrm{d}\Gamma(x) + \frac{\mu(1+\nu)\alpha}{(1-2\nu)}(\theta(x_c', \tau_F) + \theta(x_c'', \tau_F))\delta_{ij}$$

$$= \displaystyle{-}\!\!\!\!\!\!\int_\Gamma U_{kij}(x_c', x) t_k(x, \tau_F)\, \mathrm{d}\Gamma(x) + \int_\Gamma \int_{\tau_o}^{\tau_F} G_{ij}(x_c', x, \tau_F, \tau) q(x, \tau)\, \mathrm{d}\tau\, \mathrm{d}\Gamma(x). \quad (26)$$

The traction equation for a point x_c' is obtained by multiplying eqn (26) by the normal $n_j(x_c')$ and using $n_j(x_c') = -n_j(x_c'')$; it is

$$\frac{t_i(x_c', \tau_F)}{2} - \frac{t_i(x_c'', \tau_F)}{2} + n_j(x_c') \displaystyle{=}\!\!\!\!\!\!\int_\Gamma T_{kij}(x_c', x) u_k(x, \tau_F)\, \mathrm{d}\Gamma(x)$$

$$+ n_j(x_c') \left[\displaystyle{-}\!\!\!\!\!\!\int_\Gamma \int_{\tau_o}^{\tau_F} F_{ij}(x_c', x, \tau_F, \tau)\theta(x, \tau)\, \mathrm{d}\tau\, \mathrm{d}\Gamma(x) \right.$$

$$\left. + \frac{\mu(1+\nu)\alpha}{(1-2\nu)} \{\theta(x_c', \tau_F) + \theta(x_c'', \tau_F)\}\delta_{ij} \right]$$

$$= n_j(x_c') \left[\displaystyle{-}\!\!\!\!\!\!\int_\Gamma U_{kij}(x_c', x) t_k(x, \tau_F)\, \mathrm{d}\Gamma(x) + \int_\Gamma \int_{\tau_o}^{\tau_F} G_{ij}(x_c', x, \tau_F, \tau) q(x, \tau)\, \mathrm{d}\tau\, \mathrm{d}\Gamma(x) \right]. \quad (27)$$

NUMERICAL IMPLEMENTATION

The numerical implementation for crack problems in transient thermoelasticity requires both time integration and space integration. In the present implementation of the method analytical integration of the time integrals is done first. The order of the singularities of the kernel functions can be seen after the integration over time. The order of the singularities in the kernels of both the thermal equations and the elastic equations are the same as in steady state thermoelasticity (Prasad *et al.*, 1994a).

In this section the modelling strategy of the dual boundary element method is used, and discretization of the boundary integral equations is discussed. The procedures for analytical integration of the time integrals and numerical integration of the singular and nonsingular spatial integrals are presented.

Modelling strategy

As stated earlier the dual boundary element method is used to model the transient thermoelastic crack problems considered here. This formulation removes the problem of a singular matrix system that would have arisen in the final system of equations if the same set of equations were used on both crack faces. The two sets of equations for thermoelasticity are for temperature and displacement, and for flux and traction, as detailed in previous sections. The traction and the flux equations contain hypersingular integrands, which arise from the coefficients of displacement and temperature; they are of $O(1/r^2)$. Therefore the displacement and temperature themselves need to have a higher order continuity at the collocation points (nodes) than the strongly singular coefficients. This is achieved in the DBEM by using straight, quadratic discontinuous elements where hypersingular equations are applied. The modelling strategy is the same as in steady state thermoelasticity as shown in Prasad *et al.* (1994a) and can be summarized as follows:

—Temperature and displacement equations are collocated on one crack surface and flux and traction equations are collocated on the other. Since the displacement and temperature in the traction and flux equations require higher order continuity at the singular points, crack boundaries are modelled with discontinuous quadratic elements. Nodes on the discontinuous elements are at 1/6, 1/2 and 5/6 of the element length.

—Temperature and displacement equations are used for collocation at all boundary points that are not on the crack. Continuous quadratic elements are used along the non-crack boundary, except at the intersection between a crack and an edge where a discontinuous element is used.

Discretization

In all the collocations of the temperature, flux, displacement and traction boundary integral equations, the time integration of the kernels is done before the spatial integration. The temperature eqn (10) is discretized as follows:

$$c(x')(\theta(x',\tau_F)-\theta(x',\tau_o))-\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^bQ\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]\theta^{abnf}$$
$$=-\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^b\Theta\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]q^{abnf}$$
$$-\sum_{n=1}^{N}\left[\left\{\int_{\Gamma_n}N^aQ^o\,\mathrm{d}\Gamma\right\}\theta^{an}(x,\tau_o)-\left\{\int_{\Gamma_n}N^a\Theta^o\,\mathrm{d}\Gamma\right\}q^{an}(x,\tau_o)\right], \quad (28)$$

where N^a and M^b are spatial and temporal shape functions respectively. Superscripts a and b represent the spatial and temporal node numbers in each element. In the present implementation quadratic elements are used to model spatial boundary variables ($a = 1, 2, 3$) and linear ($b = 1, 2$) time interpolation is used to model the time domain. The boundary of the domain Γ is divided into N elements and the time domain is divided into F equal time steps.

Discretization of the flux eqn (17) is similar to that for the temperature equation: the discretized equation for a point x_c' on the crack surface is

$$\tfrac{1}{2}(q(x_c',\tau_F)-q(x_c',\tau_o))-\tfrac{1}{2}(q(x_c'',\tau_F)-q(x_c'',\tau_o))$$
$$+n_i(x_c')\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^bQ_i\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]\theta^{abnf}$$
$$=n_i(x_c')\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^b\Theta_i\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]q^{abnf}$$
$$+n_i(x_c')\sum_{n=1}^{N}\left[\left\{\int_{\Gamma_n}N^aQ_i^o\,\mathrm{d}\Gamma\right\}\theta^{an}(x,\tau_o)-\left\{\int_{\Gamma_n}N^a\Theta_i^o\,\mathrm{d}\Gamma\right\}q^{an}(x,\tau_o)\right]. \quad (29)$$

Similarly, the discretization of the displacement and traction eqns (18, 27) are as follows:

$$c_{ij}(x')u_j(x',\tau_F)+\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^aT_{ij}\,\mathrm{d}\Gamma\right]u_j^{an}+\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^bF_i\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]\theta^{abnf}$$
$$=\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^aU_{ij}\,\mathrm{d}\Gamma\right]t_j^{an}+\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^bG_i\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]q^{abnf}; \quad (30)$$

and

$$\frac{1}{2}(t_i(x'_c,\tau_F)-t_i(x''_c,\tau_F))+n_j(x'_c)\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a T_{kij}\,\mathrm{d}\Gamma\right]u_k^{an}$$

$$+n_j(x'_c)\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^b F_{ij}\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]\theta^{abnf}+\frac{\mu(1+\nu)a}{(1-2\nu)}(\theta(x'_c,\tau_F)+\theta(x''_c,\tau_F))$$

$$=n_j(x'_c)\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a U_{kij}\,\mathrm{d}\Gamma\right]t_k^{an}+n_j(x'_c)\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\left\{\sum_{f=1}^{F}\int_{\tau_{f-1}}^{\tau_f}M^b G_{ij}\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma\right]q^{abnf}. \quad (31)$$

In the above discretization of the boundary integral equations, different integrals can have different numbers of spatial and temporal points (nodes).

Time integration

In the present study all the time integrations of the boundary integral equations are evaluated analytically. Both constant and linear time interpolations are incorporated in the computer program. Linear time interpolations must be used for the time integration of the flux boundary integral equation, as the error in modelling using constant time interpolation is large (Prasad *et al.*, 1994b).

It can be seen from the fundamental solutions in Appendix A that the primary time variable is $(\tau_F-\tau)$ where τ_F is the time at which the results are required, and τ is the integration variable which varies from τ_o to τ_F. For the temporal integration the time $(\tau_F-\tau_o)$ is divided into F time steps of $\Delta\tau$ each $(\tau_F=\tau_o+F\Delta\tau)$.

For linear time interpolation, the temporal shape functions are given by

$$b=1 \quad \text{and} \quad M^1=\frac{\tau_f-\tau}{\Delta\tau},$$

and

$$b=2 \quad \text{and} \quad M^2=\frac{\tau-\tau_{f-1}}{\Delta\tau},$$

where $\tau_f-\tau_{f-1}=\Delta\tau$ and $\tau_{f-1}\leqslant\tau\leqslant\tau_f$.

To calculate the unknown boundary values at τ_F, the boundary values at all the previous times $(\tau_f, f=1, F-1)$ must be known. Since the time is divided into equal time steps of $\Delta\tau$, only one new set of matrices needs to be calculated in order to obtain results at τ_F; the matrices calculated at all the previous time steps are used again. To solve the final system of equations arising from the boundary integral equations, "LU" decomposition of the matrix (lower and upper) is used. By using this kind of solver, the decomposition need be done only once and the same matrix can be used at all the time steps.

The following is a typical example of the time progression scheme for the temperature (28) at time t_F:

$$c(x')(\theta(x',\tau_F)-\theta(x',\tau_o))-\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\int_{\tau_{F-1}}^{\tau_F}M^b Q(x',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma\right]\theta^{abnF}$$

$$=-\sum_{n=1}^{N}\left[\int_{\Gamma_n}N^a\int_{\tau_{F-1}}^{\tau_F}M^b\Theta(x',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma\right]q^{abnF}$$

$$+\sum_{n=1}^{N}\left[\left\{\int_{\Gamma_n}N^a Q^o\,\mathrm{d}\Gamma\right\}\theta^{an}-\left\{\int_{\Gamma_n}N^a\Theta^o\,\mathrm{d}\Gamma\right\}q^{an}\right]$$

$$+\sum_{n=1}^{N}\sum_{f=1}^{F-1}\left[\int_{\Gamma_n}N^a\int_{\tau_{f-1}}^{\tau_f}M^b Q(x',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma\right]\theta^{abnf}$$

$$-\sum_{n=1}^{N}\sum_{f=1}^{F-1}\left[\int_{\Gamma_n}N^a\int_{\tau_{f-1}}^{\tau_f}M^b\Theta(x',x,\tau_F,\tau)\,\mathrm{d}\tau\,\mathrm{d}\Gamma\right]q^{abnf}. \quad (32)$$

The above kernels, after time integration, are given in Appendix B.

Spatial integration

The order of the singularities in the boundary integral equations is summarized in Table 1.

In the present implementation, regular Gaussian quadrature is used when the collocation point is not part of an element (that is, $r \neq 0$ anywhere on the element). If the collocation point is on the element being integrated ($r = 0$ at that point), different evaluation techniques are used depending on the order of singularity and the type of element.

For the temperature boundary integral equation, the kernels of the temperature and flux are weakly singular as in the steady-state temperature equation. For the collocation point itself, it is not possible to calculate the coefficient of the temperature in the temperature equation, from rigid body motion, as in the steady-state case. The coefficient of temperature is therefore calculated exactly using the method proposed by Aliabadi and Hall (1989). The kernel of the flux is of $O(\ln r)$ and the integral is calculated by using a mixture of regular Gaussian quadrature and logarithmic Gaussian quadrature (Aliabadi and Rooke, 1991).

The flux equation is used for collocation points (nodes) on just one of the crack surfaces. Since, all the crack elements are straight and discontinuous, the continuity required by the hypersingular flux equation at singular points is automatically satisfied. Coefficients of the flux and temperature for the singular element are calculated by analytical integration. It can be seen from Appendix B that there are singular and nonsingular parts in the coefficients after time integration. The nonsingular parts are calculated with regular Gauss quadrature. There are two integrals to consider, one hypersingular (the coefficient of temperature) and one strongly singular (the coefficient of flux) : they are

$$\int_{\Gamma_n} N^a \frac{e^{-\frac{r^2}{4\kappa(\tau_F-\tau)}}}{r^2} \mathrm{d}\Gamma \quad \text{and} \quad \int_{\Gamma_n} N^a \frac{e^{-\frac{r^2}{4\kappa(\tau_F-\tau)}}}{r} \mathrm{d}\Gamma. \tag{33}$$

If the global coordinates are converted to local coordinates, the following relationships hold :

$$r = \frac{(\xi-\xi')l}{2}, \quad \mathrm{d}\Gamma = \frac{l}{2}\mathrm{d}\xi$$

and

$$\frac{r^2}{4\kappa(\tau_F-\tau)} = \beta(\xi-\xi')^2 \quad \text{where} \quad \beta = \frac{l^2}{16\kappa(\tau_F-\tau)}; \tag{34}$$

Table 1. Order of singularities in the boundary integral equations

	Coefficients of temperature	flux	displacement	traction
Temperature eqn	$\frac{r_{,k}n_k}{r}$	$\ln r$		
Flux eqn	$\frac{1}{r^2}$	$\frac{1}{r}$		
Displacement eqn	$\ln r$	not singular	$\frac{1}{r}$	$\ln r$
Traction eqn	$\frac{1}{r}$	$\ln r$	$\frac{1}{r^2}$	$\frac{1}{r}$

l is the length of the element, ξ is the local coordinate of the element $(-1 \leqslant \xi \leqslant +1)$ and ξ' is the local coordinate of the collocation point. Substitution of the above relations allows the singular integrals to be written as follows:

$$\frac{2}{l}\int_{-1}^{+1} N^a \frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')^2}\,\mathrm{d}\xi \quad \text{and} \quad \int_{-1}^{+1} N^a \frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')}\,\mathrm{d}\xi. \tag{35}$$

The shape functions N^a can be factorized as

$$N^a = A^a(\xi-\xi')^2 + B^a(\xi-\xi') + C^a. \tag{36}$$

After substitution of the shape functions, the strongly singular integral can be written as

$$\int_{-1}^{1} N^a \frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')}\,\mathrm{d}\xi = A^a\int_{-1}^{1}(\xi-\xi')e^{-\beta(\xi-\xi')^2}\,\mathrm{d}\xi + B^a\int_{-1}^{1} e^{-\beta(\xi-\xi')^2}\,\mathrm{d}\xi + C^a\int_{-1}^{1}\frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')}\,\mathrm{d}\xi; \tag{37}$$

and the hypersingular integral as

$$\int_{-1}^{1} N^a \frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')^2}\,\mathrm{d}\xi = A^a\int_{-1}^{1} e^{-\beta(\xi-\xi')^2}\,\mathrm{d}\xi + B^a\int_{-1}^{1}\frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')}\,\mathrm{d}\xi + C^a\int_{-1}^{1}\frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')^2}\,\mathrm{d}\xi. \tag{38}$$

The nonsingular integrals can be calculated numerically, and the singular terms analytically as follows:

$$\int_{-1}^{1}\frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')}\,\mathrm{d}\xi = \tfrac{1}{2}[E_1(\beta(1+\xi')^2) - E_1(\beta(1-\xi')^2)] \tag{39}$$

and

$$\int_{-1}^{1}\frac{e^{-\beta(\xi-\xi')^2}}{(\xi-\xi')^2}\,\mathrm{d}\xi = -\left[\frac{e^{-\beta(1-\xi')^2}}{(1-\xi')} + \frac{e^{-\beta(1+\xi')^2}}{(1+\xi')}\right] - 2\beta\int_{-1}^{1} e^{-\beta(\xi-\xi')^2}\,\mathrm{d}\xi, \tag{40}$$

where the exponential integral is defined as $E_1(x) = \int_x^\infty (e^{-s}/s)\,\mathrm{d}s$.

In both the displacement equation and the traction equation, the coefficients of displacement and traction are the same as in elastostatics (Portela *et al.*, 1992) and steady-state thermoelasticity (Prasad *et al.*, 1994a), and are evaluated in the same way.

In the displacement equation the coefficient of flux is not singular, so it can be calculated using regular Gauss quadrature; the coefficient of temperature is singular of $O(\ln r)$ and is calculated as before. In the traction equation, the coefficient of flux is also of $O(\ln r)$ singular; the coefficient of temperature is of $O(1/r)$, and is evaluated from (37).

In the evaluation of the singularities in the kernels it should be observed that:

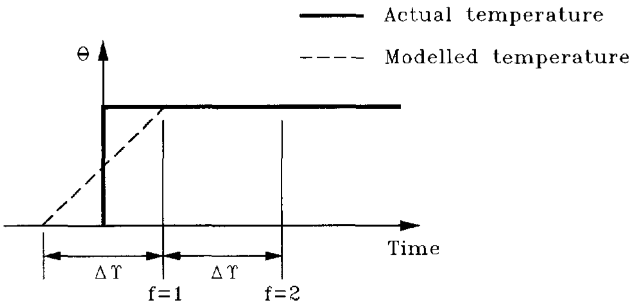

Fig. 1. Thermal shock modelling for linear time integration.

$$\lim_{r\to 0}\left\{1-e^{-\frac{r^2}{4\kappa\tau}}\right\} = O(r^2)$$

and

$$\lim_{x\to 0}\{E_1(x)\} = O(\ln x).$$

In the present work, the thermal and the elastic equations are solved together as shown by Prasad *et al.*, (1994a). This procedure is also convenient for modelling physically coupled problems because both thermal and elastic problems can be solved simultaneously. The two equations at time τ_F can be written together in matrix form as follows:

$$\begin{bmatrix} u_1^F, & t_1^F, \\ u_2^F, & t_2^F \\ \theta^F, & q^F \end{bmatrix} + \begin{bmatrix} H_{11}^{(u,t)} & H_{12}^{(u,t)} & H_{1\theta}^{(u,t)F} \\ H_{12}^{(u,t)} & H_{22}^{(u,t)} & H_{2\theta}^{(u,t)F} \\ 0 & 0 & H_{\theta\theta}^{(u,t)F} \end{bmatrix} \begin{bmatrix} u_1^F \\ u_2^F \\ \theta^F \end{bmatrix}$$
$$= \begin{bmatrix} G_{11}^{(u,t)} & G_{12}^{(u,t)} & G_{1\theta}^{(u,t)F} \\ G_{12}^{(u,t)} & G_{22}^{(u,t)} & G_{2\theta}^{(u,t)F} \\ 0 & 0 & G_{\theta\theta}^{(u,t)F} \end{bmatrix} \begin{bmatrix} t_1^F \\ t_2^F \\ q^F \end{bmatrix} + \sum_{f=1}^{F-1} \left\{ \begin{bmatrix} G_{1\theta}^{(u,t)f} \\ G_{2\theta}^{(u,t)f} \\ G_{\theta\theta}^{(u,t)f} \end{bmatrix} [q^f] - \begin{bmatrix} H_{1\theta}^{(u,t)f} \\ H_{2\theta}^{(u,t)f} \\ H_{\theta\theta}^{(u,t)f} \end{bmatrix} [\theta^f] \right\} \quad (41)$$

If the initial conditions are not homogeneous, there will be additional terms. Superscripts (u, t) mean that the H and G coefficients belong to either displacement and temperature equations, or traction and flux equations, depending on the location of the collocation point. θ^f and q^f for $f < F$ are known values. The matrices $[H_{i\theta}^{(u,t),f}]$ and $[G_{i\theta}^{(u,t),f}]$ (where $i = 1$, 2, θ) for $f > 2$ have been calculated and stored from previous time steps. If the problem and number of time steps are large the storage space required can be quite substantial. The results for single and double precision calculations are very close for initial time steps but as the number of time steps increase the errors for single precision increase. So, double precision calculations are used in the numerical implementations reported here.

If the problem being modelled is a thermal shock problem (a sudden rise in temperature) the flux is infinite at the boundary where the thermal shock is applied. This is a problem if the time interpolation is linear. This difficulty is overcome in the present study by approximating the first step as a linear increase in temperature (Zienkiewicz, 1989), as shown in Fig. 1.

CALCULATION OF STRESS INTENSITY FACTORS

The magnitude of the stress intensity factor is a measure of the severity of the crack in both dynamic and static problems. There are many methods of calculating stress intensity factors; one of the most accurate ways is via the path independent $\hat{J}$-integral (Aliabadi and Rooke, 1991). In the present work the $\hat{J}$-integral for thermal and mechanical conditions, defined by Kishimoto *et al.* (1980), is used to calculate the stress intensity factors K_I and K_{II}. It is given by

$$\hat{J} = \hat{J}_1 = \int_S \left(W_e n_1 - t_i \frac{\partial u_i}{\partial x'_1} \right) \mathrm{d}s + \int_A \alpha \sigma_{ii} \frac{\partial \theta}{\partial x'_1} \mathrm{d}A, \tag{42}$$

where

$$\hat{J}_1 = \frac{K_I^2 + K_{II}^2}{E'}; \tag{43}$$

W_e is the elastic strain energy; $E' = E$ for plane stress conditions and $E' = E/(1-\nu^2)$ for plane strain conditions; S is the $\hat{J}$ integral contour and A is the area enclosed by the contour. The elastic strain energy W_e is given by

$$W_e = \int_0^{\varepsilon_{ij}^e} \sigma_{ij} \, \mathrm{d}\varepsilon_{ij}^e \quad \text{where} \quad \varepsilon_{ij}^e = \varepsilon_{ij} - \alpha\theta\delta_{ij}. \tag{44}$$

The method of evaluation of the $\hat{J}$-integral in transient thermoelasticity is the same as in steady-state thermoelasticity (Prasad *et al.*, 1994a). The contour integral is evaluated around a circular path from any node on the crack surface to the node on the opposite crack surface, with the crack tip as centre. The circular path is divided into linear segments and the region inside the circle is divided into triangular segments; each triangle has an apex at the crack tip and the other two are on the circular path. The surface integral and domain integral can be calculated with Gauss quadrature. Care must be taken in the evaluation of the domain integral as stresses are singular at the crack tip. The singularity of the stress can be eliminated by transforming the triangular segments to square segments as shown in Aliabadi and Rooke (1991). The Jacobian of this transformation cancels the weak singularity of the stress, and regular Gauss quadrature can then be used over the transformed square region.

In eqn (43) K_I and K_{II} are coupled; they can be uncoupled, to symmetric and unsymmetric components, as shown by Portela *et al.* (1992) and Prasad *et al.* (1994a). The uncoupled K_I and K_{II} can then be calculated as follows,

$$\hat{J}_1 = \hat{J}_1^I + \hat{J}_1^{II},$$

where

$$\frac{K_I^2}{E'} = \hat{J}_1^I \quad \text{and} \quad \frac{K_{II}^2}{E'} = \hat{J}_1^{II}. \tag{45}$$

The uncoupled terms $\hat{J}_1^I$ and $\hat{J}_1^{II}$ can be calculated by the substitution of symmetric and antisymmetric components respectively, into eqn (42).

The calculation of $\hat{J}$-integrals requires the evaluation of internal values at the Gauss points of the linear segment of the contour and the triangular segment of the domain. The internal values required are of stress, strain, temperature and derivatives of displacement and temperature. The interior boundary integral formulations for these internal values are given in the previous sections. In transient thermoelasticity, these internal values are required at all the time steps. Since the time step ($\Delta\tau$) is constant, calculation of the values at the current time step can be expressed in terms of matrices of previous steps. The storage of these matrices requires considerable computer space even when only the necessary internal values are calculated at each internal point. It should be noted that multiple crack problems would require additional storage as the internal point locations are different.

NUMERICAL RESULTS

Numerical results for three crack problems with transient thermal boundary conditions are presented in this section. The stress intensity factors were calculated from $\hat{J}$-integrals on different circular paths; each path is referred to by a path number. The number is the crack node number where the path starts. The node numbers increase, counting from unity at the crack tip. Path independence was checked and all the results quoted refer to path 5.

All the examples analyzed have the same material properties: Young's modulus $E = 2.184 \cdot 10^{-5}$ Pa; Poisson's ratio $\nu = 0.3$; the coefficient of linear expansion $\alpha = 1.67 \cdot 10^{-5}$ per °C and the coefficient of diffusion $\kappa = 1.0$ m²/s. The variation of the stress intensity factors with time is presented in non-dimensional form. All the problems are solved in plain strain conditions.

Rectangular plate with a central crack

A rectangular plate of width $2W$, length $2L$ and a central crack of length $2a$ is shown in Fig. 2a. This configuration with $L/W = 1.0$ is solved for two different sets of boundary conditions representing pure mode *I* and pure mode *II* respectively. The results are compared with the finite element results calculated by Emmel and Stamm (1985). Initial conditions for both boundary condition sets are zero temperature and zero flux.

1. Boundary condition I (Pure mode I)
   $\theta(\tau) = 0$ on the crack
   $\theta(\tau \geqslant 0) = \theta_1$ around the boundary
2. Boundary condition II (Pure mode II)
   $q(\tau) = 0$ on the crack
   $q(\tau) = 0, x = \pm W, |y| < L$
   $\theta(\tau \geqslant 0) = \pm\theta_1, |x| \leqslant W, y = \pm L$

Because of symmetry only half the problem needs to be considered. That is, since $q = 0$ along the line $x = 0$ for both boundary condition sets, the problem can be modelled as in Fig. 2b. The geometry is modelled using 34 quadratic elements with six elements on each crack surface. Both sets of boundary conditions represent a problem of thermal shock.

When $\theta_1 = 1$°C and $a/W = 0.5$ the results obtained can be compared with those of Emmel and Stamm (1985). It can be seen from Fig. 3 that the present boundary element results compare well with the finite element results. The size of the time step has little effect on the result after the first few time steps in the case of linear time interpolation. For constant interpolation, results differ for longer time. So, in all the subsequent calculations only linear time interpolation is used. In Fig. 4 values of K_I for various a/W are shown. The stress intensity factors in Figs 3, 4 and 5 are normalized with a factor of $F = \alpha(\theta_1)E(W)^{0.5}$. Time is normalized as $\Upsilon = \kappa\tau/W^2$. Results are shown for three different

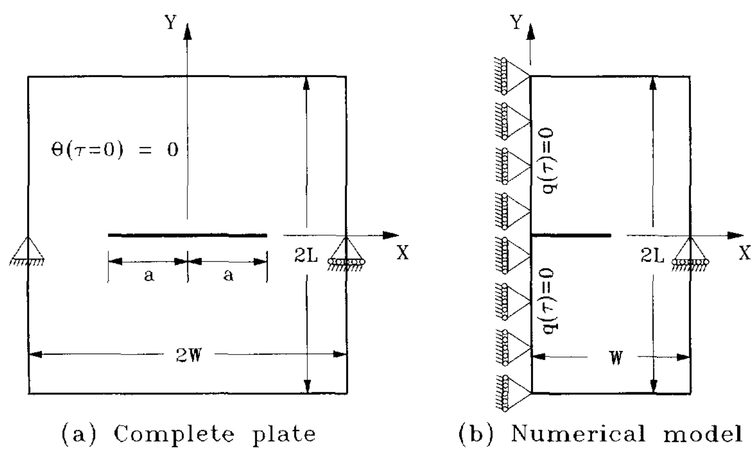

Fig. 2. Rectangular plate with center crack.

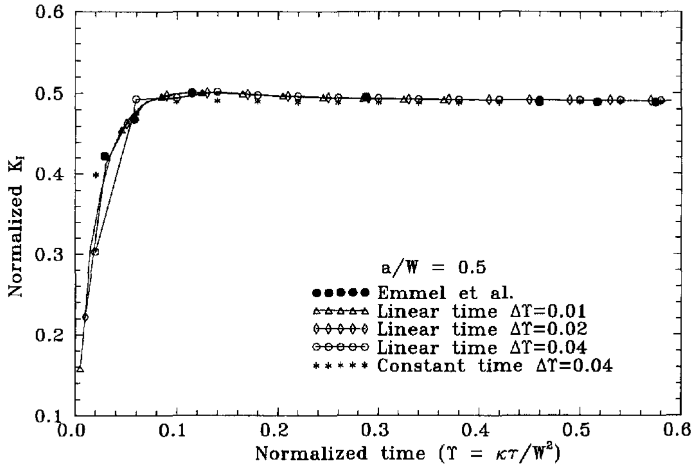

Fig. 3. Comparison of results between different time steps.

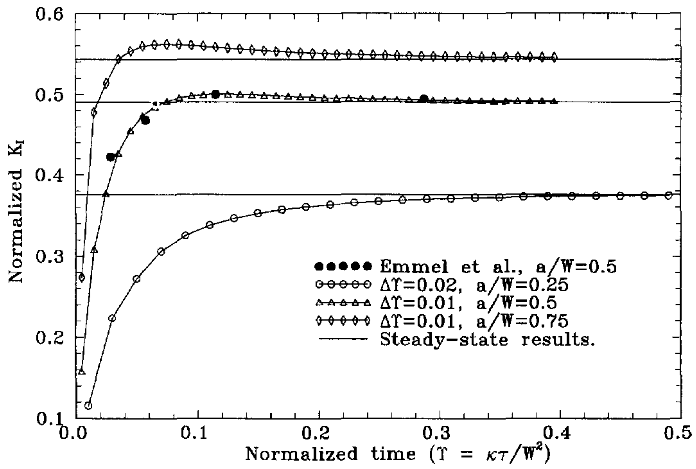

Fig. 4. Normalized K_I values for different ratios of a/W of center crack.

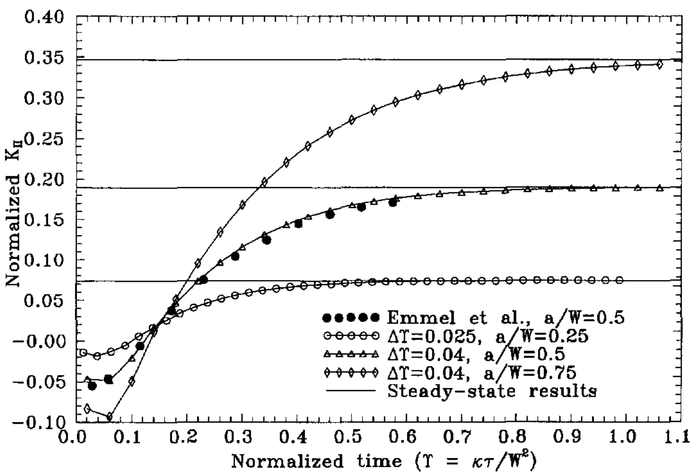

Fig. 5. Normalized K_{II} values for different ratios of a/W of center crack.

crack lengths. Again the results compare well with those of Emmel and Stamm (1985) for $a/W = 0.5$. It can be seen that the transient results tend to the steady state values.

The values for K_{II} start negative and become positive. This is because the effect of temperature on the crack is minimal initially, but it increases with time. If the temperature

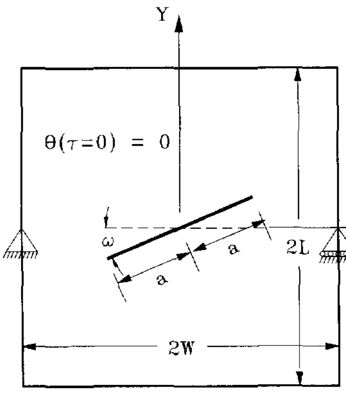

Fig. 6. Rectangular plate with an inclined crack.

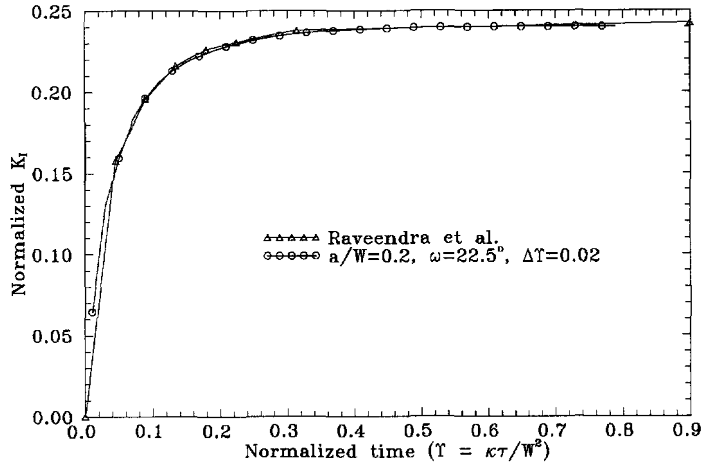

Fig. 7. Normalized K_I values for inclined crack.

on the top edge ($y = H$) is $+\theta_1$ and on the bottom edge ($y = -H$) is $-\theta_1$ the final K_{II} is negative as the top crack face expands and the bottom face contracts. However, initially the effect is the opposite, similar to a pure mode *II* problem with mechanical boundary conditions.

Rectangular plate with an inclined crack

A rectangular plate of width $2W$, length $2L$ with a center crack of length $2a$ at an angle ω is shown in Fig. 6. The configuration with $L/W = 1.0$ and $a/W = 0.2$ is solved for one set of boundary conditions which gives mixed mode behavior. Initial boundary conditions are zero temperature and zero flux. The temperature boundary conditions are θ_1 on the outer boundary and θ_2 on the crack. The geometry is modelled using 32 elements with six elements on each crack surface. The number of elements used on the crack are same as the number used by Raveendra and Banerjee (1992).

The stress intensity factors are normalized with $E\alpha(\theta_1 - \theta_2)\sqrt{W}/(1-\nu)$ and time is normalized as $\Upsilon = \kappa\tau/W^2$. The results are shown in Figs (7) and (8), as a function of time, and compare very well with the results from Raveendra and Banerjee (1992) who used a boundary element analysis.

CONCLUSIONS

The two-dimensional dual boundary element method has been applied to the analysis of transient thermoelastic problems in cracked structures. A distinct set of equations was obtained for the two different crack surfaces by applying temperature and displacement equations when collocating on one crack surface and flux and traction equations when

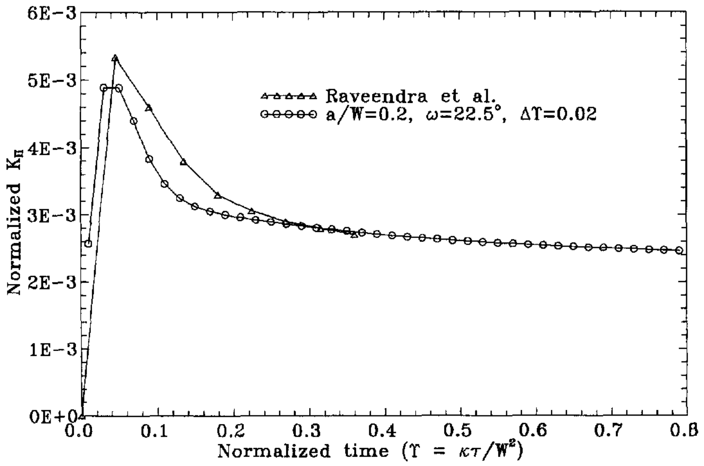

Fig. 8. Normalized K_{II} values for inclined crack.

collocating on the other. The body is considered to be in a steady-state condition, before the transient boundary conditions are applied; this leads to a boundary only formulation. Time integration of the kernels in temperature, displacement and traction equations is done by linear time integration. The time integration of kernels in the flux equation must be done by linear time integration as the errors induced by using constant time integration are high. The spatial integration is done by using quadratic elements all over the boundary, discontinuous on the crack and continuous on rest of the boundary. The method presented allows general two-dimensional, mixed-mode problems to be solved in an efficient and general single-region formulation. Stress intensity factors over time were calculated using a $\hat{J}$ integral technique. The results from rectangular plates with a central crack were compared with existing finite element results and found to be in good agreement. It has been demonstrated that the results are accurate and that the dual boundary element method can be applied to any two-dimensional geometry with transient thermoelastic boundary conditions.

Acknowledgement—The first two authors acknowledge the support of European Commission Esprit project 6874, Microsystem Analysis and Simulation System (MASS).

REFERENCES

Aliabadi, M. H. and Hall, W. S. (1989). Two-dimensional boundary element kernel integration using series expansions. *Engng Anal. Boundary Elements* **6**, 140–143.

Aliabadi, M. H. and Rooke, D. P. (1991). *Numerical Fracture Mechanics*, Computational Mechanics Publications, Southampton.

Blandford, G. E., Ingraffea, A. R. and Liggett, J. A. (1981). Two-dimensional stress intensity factor computations using the boundary element method. *Int. J. Num. Meth. Engng* **17**, 387–404.

Brebbia, C. A., Telles, J. and Wrobel, L. (1984). *Boundary Element Techniques: Theory and Applications in Engineering*, Springer-Verlag, New York.

Carslaw, H. S. and Jaeger, J. C. (1959). *Conduction of Heat in Solids*, 2nd ed., Clarendon Press, Oxford.

Cruse, T. A. (1977). Mathematical foundations of the boundary integral equation method in solid mechanics. Report No. AFOSR-TR-77-1002, Pratt and Whitney Aircraft Group.

Dargush, G. F. and Banerjee, P. K. (1989). Development of a boundary element method for time-dependent planar thermoelasticity. *Int. J. Solids Struct.* **25**, 999–1021.

El-Zafrany, A. (1993). *Techniques of the Boundary Element Method*, Ellis Horwood, London.

Emery, A. F., Neighbors, P. K., Kobayashi, A. S. and Love, W. J. (1977). Stress intensity factors in edge-cracked plates subjected to transient thermal singularities. *J. Pres. Ves. Tech.* **99**, 100–105.

Emmel, E. and Stamm, H. (1985). Calculation of stress intensity factors of thermally loaded cracks using the finite element method. *Int. J. Pres. Ves. Piping* **19**, 1–17.

Hellen, T. K., Cesari, F. and Maitan, A. (1982). The application of fracture mechanics in thermally stressed structures. *Int. J. Pres. Ves. Piping* **10**, 181–204.

Ionescu-Cazimir, V. (1964). Problem of linear coupled thermoelasticity. Theorems on reciprocity for the dynamic problem of coupled thermoelasticity. *I. Bull. Acad. Polonaise Sci. Series Sci. Tech.* **12**, 473–488.

Kishimoto, K., Aoki, S. and Sakata, M. (1980). On the path independent integral-$\hat{J}$. *Engng Fract. Mech.* **13**, 841–850.

Krishnasamy, G., Rizzo, F. J. and Rudolphi, T. J. (1992). Continuity requirements for density functions in the boundary integral equation method. *Computational Mech.* **9,** 267–284.

Portela, A., Aliabadi, M. H. and Rooke, D. P. (1992). The dual boundary element method: effective implementation for crack problems. *Int. J. Num. Meth. Engng* **33,** 1269–1287.

Prasad, N. N. V., Aliabadi, M. H. and Rooke, D. P. (1994a). The dual boundary element method for thermoelastic crack problems. *Int. J. Fracture* **66,** 255–272.

Prasad, N. N. V., Aliabadi, M. H. and Rooke, D. P. (1994b). On the use of the hypersingular boundary element diffusion equation. *Boundary Element Comm.* **5,** 215–217.

Raveendra, S. T. and Banerjee, P. K. (1992). Boundary element analysis of cracks in thermally stressed planar structures. *Int. J. Solids Struct.* **29,** 2301–2317.

Sládek, V. and Sládek, J. (1989). Computation of thermal stresses in quasi-static non-stationary thermoelasticity using boundary elements. *Int. J. Num. Meth. Engng* **28,** 1131–1144.

Sládek, V. and Sládek, J. (1992). Advanced thermoelastic analysis. In *Boundary Element Methods in Heat Transfer*, (eds Wrobel, L. C. and Brebbia, C. A.). Computational Mechanics Publications, Southampton, Elsevier Applied Science, London.

Tanaka, M., Togoh, H. and Kikutu, M. (1984a). Boundary element method applied to 2-D thermoelastic problems in steady and non-steady states. *Engng Anal.* **1,** 13–19.

Tanaka, M., Togoh, H. and Kikuta, M. (1984b). Fracture mechanics application in thermoelastic states. In *Topics in Boundary Element Research*, Vol. 1 (ed Brebbia, C. A.). Springer-Verlag, Berlin, pp. 59–77.

Zienkiewicz, O. C. (1989). *The Finite Element Method*, McGraw-Hill, London.

APPENDIX A: FUNDAMENTAL SOLUTIONS

The fundamental solutions of the temperature equation (Brebbia *et al.*, 1984) are expressed in terms of the following:

$$s = \frac{r^2}{4\kappa(\tau_F-\tau)}, \quad r_{,i} = \frac{r_i}{r} \quad \text{and} \quad \kappa = \frac{\lambda}{c_\varepsilon};$$

where $r = |x'-x|$, λ is thermal conductivity and c_ε is specific heat at constant strain. Thus

$$\Theta(x', x, \tau_F, \tau) = \frac{1}{4\pi\lambda(\tau_F-\tau)} e^{-s} \tag{A.1}$$

and

$$Q(x', x, \tau_F, \tau) = \frac{r_{,k} n_k r}{8\pi\kappa(\tau_F-\tau)^2} e^{-s}. \tag{A.2}$$

The flux eqn (17) fundamental solutions are written as follows

$$\Theta_i(x', x, \tau_F, \tau) = \lambda \frac{\partial \Theta(x', x, \tau_F, \tau)}{\partial x'_i} = \frac{r r_{,i}}{8\pi\kappa(\tau_F-\tau)^2} e^{-s} \tag{A.3}$$

$$Q_i(x', x, \tau_F, \tau) = \lambda \frac{\partial Q(x', x, \tau_F, \tau)}{\partial x'_i} = \left[-\frac{n_i c_\varepsilon}{8\pi(\tau_F-\tau)^2} + \frac{r_{,k} n_k r_{,i} r^2 c_\varepsilon}{16\pi\kappa(\tau_F-\tau)^3} \right] e^{-s} \tag{A.4}$$

The Kelvin fundamental solutions (Brebbia *et al.* 1984) for the displacement equation are written as follows:

$$U_{ij}(x', x) = \frac{(1+\nu)}{4\pi E(1-\nu)} [(4\nu-3)\ln r\,\delta_{ij} + r_{,i} r_{,j}] \tag{A.5}$$

$$T_{ij}(x', x) = \frac{-1}{4\pi(1-\nu)r} \left\{ [(1-2\nu)\delta_{ij} + 2 r_{,i} r_{,j}] \frac{\partial r}{\partial n} - (1-2\nu)(r_{,i} n_j - r_{,j} n_i) \right\} \tag{A.6}$$

$$F_i(x', x, \tau_F, \tau) = \frac{(1+\nu)\alpha}{4\pi(1-\nu)} \left[\frac{2\kappa}{r^2}(2 r_{,i} r_{,k} n_k - n_i)(1-e^{-s}) - \frac{r_{,i} r_{,k} n_k}{(\tau_F-\tau)} e^{-s} \right] \tag{A.7}$$

$$G_i(x', x, \tau_F, \tau) = \frac{(1+\nu)\alpha}{2\pi c_\varepsilon(1-\nu)} \frac{r_{,i}}{r} (1-e^{-s}) \tag{A.8}$$

The fundamental solutions (Aliabadi and Rooke, 1991) of the traction equation are written as follows:

$$U_{kij}(x', x) = \frac{1}{4\pi(1-\nu)r} [(1-2\nu)(\delta_{ki} r_{,j} + \delta_{kj} r_{,i} - \delta_{ij} r_{,k}) + 2 r_{,i} r_{,j} r_{,k}] \tag{A.9}$$

$$T_{kij}(x', x) = \frac{E}{4\pi(1+\nu)(1-\nu)r^2}\left\{2\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij}r_{,k}+\nu(\delta_{ik}r_{,j}+\delta_{jk}r_{,i})-4r_{,i}r_{,j}r_{,k}]\right.$$

$$\left.+2\nu(n_i r_{,j} r_{,k}+n_j r_{,k} r_{,i})+(1-2\nu)(2n_k r_{,i} r_{,j}+n_j\delta_{ik}+n_i\delta_{jk})-(1-4\nu)n_k\delta_{ij}\right\} \quad (A.10)$$

$$F_{ij}(x', x, \tau_F, \tau) = -\frac{E\alpha}{4\pi(1-\nu)}\frac{1}{r}\left[(4r_{,l}n_l r_{,i}r_{,j}-r_{,j}n_i-r_{,i}n_j-\delta_{ij}r_{,l}n_l)\left\{\frac{e^{-s}}{(\tau_F-\tau)}-\frac{4\kappa}{r^2}(1-e^{-s})\right\}\right.$$

$$\left.+\frac{r_{,l}n_l r^2}{2\kappa(\tau_F-\tau)^2}\left\{r_{,i}r_{,j}+\frac{\nu}{(1-2\nu)}\delta_{ij}\right\}e^{-s}\right] \quad (A.11)$$

$$G_{ij}(x', x, \tau_F, \tau) = -\frac{E\alpha}{4\pi\lambda(1-\nu)}\left[\frac{e^{-s}}{(\tau_F-\tau)}\left(r_{,i}r_{,j}+\frac{\nu}{(1-2\nu)}\delta_{ij}\right)-\frac{4\kappa}{r^2}\left(r_{,i}r_{,j}-\frac{\delta_{ij}}{2}\right)(1-e^{-s})\right] \quad (A.12)$$

The terms due to the initial temperature and flux in the temperature and flux equations are as follows:

$$\Theta^o(x', x, \tau_F, \tau_o) = \int_{\tau_o}^{\tau_F}\Theta(x', x, \tau_F, \tau)\,d\tau = \frac{1}{4\pi\lambda}E_1(s_o) \quad (A.13)$$

$$Q^o(x', x, \tau_F, \tau_o) = \int_{\tau_o}^{\tau_F}Q(x', x, \tau_F, \tau)\,d\tau = \frac{1}{2\pi}\frac{r_{,k}n_k}{r}e^{-s_o} \quad (A.14)$$

$$\Theta_i^o(x', x, \tau_F, \tau_o) = \frac{r_{,i}}{2\pi r}e^{-s_o} \quad (A.15)$$

$$Q_i^o(x', x, \tau_F, \tau_o) = \frac{\lambda}{2\pi}\left(-\frac{n_i}{r^2}+\frac{2r_{,k}n_k r_{,i}}{r^2}+\frac{r_{,k}n_k r_{,i}}{2\kappa\Delta\tau}\right)e^{-s_o} \quad (A.16)$$

where $s_o = r^2/(4\kappa(\tau_F-\tau_o))$ and $E_1(s_o) = \int_{s_o}^{\infty}(e^{-x}/x)\,dx$.

APPENDIX B: TIME INTERPOLATION OF FUNDAMENTAL SOLUTIONS

The linear and constant time-integration of the kernels are presented in this Appendix. The following definitions are used to simplify the equations,

$$s_f = \frac{r^2}{4\kappa(\tau_F-\tau_f)} \quad \text{and} \quad s_{f-1} = \frac{r^2}{4\kappa(\tau_F-\tau_{f-1})};$$

for constant time interpolation

$$[M^b] = 1,$$

for linear time interpolation

$$[M^b] = \left[\frac{\tau_f-\tau}{\Delta\tau}, \frac{\tau-\tau_{f-1}}{\Delta\tau}\right].$$

The constant and linear time interpolation of the temperature equation gives the following:

for constant

$$\int_{\tau_{f-1}}^{\tau_f} Q\,d\tau = Q^f = -\frac{r_{,k}n_k}{2\pi r}[e^{-s_f}-e^{-s_{f-1}}] \quad (B.1)$$

and

$$\int_{\tau_{f-1}}^{\tau_f}\Theta\,d\tau = \Theta^f = -\frac{1}{4\pi\lambda}[E_1(s_f)-E_1(s_{f-1})]; \quad (B.2)$$

for linear

$$\int_{\tau_{f-1}}^{\tau_f} QM^b\,d\tau = \begin{bmatrix} -(F-f)Q^f-\dfrac{r_{,k}n_k r}{8\pi\kappa\Delta\tau}[E_1(s_f)-E_1(s_{f-1})] \\ (F-f+1)Q^f+\dfrac{r_{,k}n_k r}{8\pi\kappa\Delta\tau}[E_1(s_f)-E_1(s_{f-1})] \end{bmatrix} \quad (B.3)$$

and

$$\int_{\tau_{f-1}}^{\tau_f} \Theta M^h \,\mathrm{d}\tau = \begin{bmatrix} -(F-f)\Theta^f - \dfrac{r^2}{16\pi\kappa\lambda\Delta\tau}\left[\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right] + \dfrac{r^2}{16\pi\kappa\lambda\Delta\tau}[E_1(s_f) - E_1(s_{f-1})] \\ (F-f+1)\Theta^f + \dfrac{r^2}{16\pi\kappa\lambda\Delta\tau}\left[\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right] - \dfrac{r^2}{16\pi\kappa\lambda\Delta\tau}[E_1(s_f) - E_1(s_{f-1})] \end{bmatrix}. \tag{B.4}$$

The constant and linear time interpolations of the flux equation gives the following:
for constant

$$\int_{\tau_{f-1}}^{\tau_f} Q_i \,\mathrm{d}\tau = Q_i^f = \frac{\lambda}{\pi r^2}\left(\frac{n_i}{2} - r_{,k}n_k r_{,i}\right)(e^{-s_f} - e^{-s_{f-1}}) - \frac{r_{,k}n_k r_{,i}}{4\pi} c_\varepsilon\left(\frac{e^{-s_f}}{(\tau_F - \tau_f)} - \frac{e^{-s_{f-1}}}{(\tau_F - \tau_{f-1})}\right) \tag{B.5}$$

and

$$\int_{\tau_{f-1}}^{\tau_f} \Theta_i \,\mathrm{d}\tau = \Theta_i^f = \frac{r_{,i}}{2\pi r}(-e^{-s_f} + e^{-s_{f-1}}); \tag{B.6}$$

for linear

$$\int_{\tau_{f-1}}^{\tau_f} Q_i M^h \,\mathrm{d}\tau = \begin{bmatrix} -(F-f)Q_i^f + \dfrac{n_i c_\varepsilon}{8\pi\Delta\tau}[E_1(s_f) - E_1(s_{f-1})] - \dfrac{r_{,k}n_k r_{,i}c_\varepsilon}{4\pi\Delta\tau}[e^{-s_f} - e^{-s_{f-1}}] \\ (F-f+1)Q_i^f - \dfrac{n_i c_\varepsilon}{8\pi\Delta\tau}[E_1(s_f) - E_1(s_{f-1})] + \dfrac{r_{,k}n_k r_{,i}c_\varepsilon}{4\pi\Delta\tau}[e^{-s_f} - e^{-s_{f-1}}] \end{bmatrix} \tag{B.7}$$

and

$$\int_{\tau_{f-1}}^{\tau_f} \Theta_i M^h \,\mathrm{d}\tau = \begin{bmatrix} -(F-f)\Theta_i^f - \dfrac{r r_{,i}}{8\pi\kappa^2\Delta\tau}[E_1(s_f) - E_1(s_{f-1})] \\ (F-f+1)\Theta_i^f + \dfrac{r r_{,i}}{8\pi\kappa^2\Delta\tau}[E_1(s_f) - E_1(s_{f-1})] \end{bmatrix}. \tag{B.8}$$

The constant and linear time interpolation of the displacement equation gives the following: for constant

$$\int_{\tau_{f-1}}^{\tau_f} F_i \,\mathrm{d}\tau = f_i = \frac{(1+\nu)\alpha}{4\pi(1-\nu)}\left[(2r_{,i}r_{,k}n_k - n_i)\left(-\frac{(1-e^{-s_f})}{s_f} + \frac{(1-e^{-s_{f-1}})}{s_{f-1}}\right) + \frac{n_i}{2}[E_1(s_f) - E_1(s_{f-1})]\right] \tag{B.9}$$

and

$$\int_{\tau_{f-1}}^{\tau_f} G_i \,\mathrm{d}\tau = g_i = \frac{(1+\nu)\alpha r_{,i} r}{4\pi\lambda(1-\nu)}\left[\left(-\frac{(1-e^{-s_f})}{s_f} + \frac{(1-e^{-s_{f-1}})}{s_{f-1}}\right) - \tfrac{1}{2}[E_1(s_f) - E_1(s_{f-1})]\right]; \tag{B.10}$$

for linear

$$\int_{\tau_{f-1}}^{\tau_f} F_i M^h \,\mathrm{d}\tau = \begin{bmatrix} -(F-f)f_i - \dfrac{(1+\nu)\alpha r^2}{32\pi(1-\nu)\kappa\Delta\tau}\left[\left(r_{,i}r_{,k}n_k - \dfrac{n_i}{2}\right)\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right)\right. \\ \left. + \left(r_{,i}r_{,k}n_k + \dfrac{n_i}{2}\right)\left\{(E_1(s_f) - E_1(s_{f-1})) - \left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right)\right\}\right] \\ (F-f+1)f_i + \dfrac{(1+\nu)\alpha r^2}{32\pi(1-\nu)\kappa\Delta\tau}\left[\left(r_{,i}r_{,k}n_k - \dfrac{n_i}{2}\right)\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right)\right. \\ \left. + \left(r_{,i}r_{,k}n_k + \dfrac{n_i}{2}\right)\left\{(E_1(s_f) - E_1(s_{f-1})) - \left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right)\right\}\right] \end{bmatrix} \tag{B.11}$$

and

$$\int_{\tau_{f-1}}^{\tau_f} G_i M^h \,\mathrm{d}\tau = \begin{bmatrix} -(F-f)g_i - \dfrac{(1+\nu)\alpha r_{,i} r^3}{64\pi(1-\nu)\kappa\lambda\Delta\tau}\left[\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right)\right. \\ \left. + \left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right) - (E_1(s_f) - E_1(s_{f-1}))\right] \\ (F-f+1)g_i + \dfrac{(1+\nu)\alpha r_{,i} r^3}{64\pi(1-\nu)\kappa\lambda\Delta\tau}\left[\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right)\right. \\ \left. + \left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right) - (E_1(s_f) - E_1(s_{f-1}))\right] \end{bmatrix}. \tag{B.12}$$

The constant and linear time interpolation of the traction equation gives the following: for constant

$$\int_{\tau_{f-1}}^{\tau_f} F_{ij}\,\mathrm{d}\tau = f_{ij} = \frac{E\alpha}{4\pi(1-\nu)r}\left[(r_{,j}n_i + r_{,i}n_j + r_{,l}n_l(\delta_{ij} - 4r_{,i}r_{,j}))\left(\frac{(1-e^{-s_f})}{s_f} - \frac{(1-e^{-s_{f-1}})}{s_{f-1}}\right) + 2r_{,l}n_l\left(r_{,i}r_{,j} + \frac{\nu\delta_{ij}}{(1-2\nu)}\right)(e^{-s_f} - e^{-s_{f-1}})\right]; \quad \text{(B.13)}$$

and

$$\int_{\tau_{f-1}}^{\tau_f} G_{ij}\,\mathrm{d}\tau = g_{ij} = \frac{E\alpha}{4\pi\kappa\lambda(1-\nu)}\left[\left(\frac{\delta_{ij}}{2} - r_{,i}r_{,j}\right)\left(\frac{(1-e^{-s_f})}{s_f} - \frac{(1-e^{-s_{f-1}})}{s_{f-1}}\right) + \frac{\delta_{ij}}{2(1-2\nu)}(E_1(s_f) - E_1(s_{f-1}))\right]. \quad \text{(B.14)}$$

For linear

$$\int_{\tau_{f-1}}^{\tau_f} F_{ij}M^b\,\mathrm{d}\tau = \begin{bmatrix} -(F-f)f_{ij} - \dfrac{E\alpha r}{32\pi(1-\nu)\kappa\Delta\tau}\Bigg[(4r_{,l}n_l r_{,i}r_{,j} - r_{,j}n_i - r_{,i}n_j - \delta_{ij}r_{,l}n_l) \\ \times\left\{\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right) - \left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right)\right\} \\ -\left(r_{,j}n_i + r_{,i}n_j + \delta_{ij}\dfrac{(1+2\nu)}{(1-2\nu)}r_{,l}n_l\right)(E_1(s_f) - E_1(s_{f-1}))\Bigg] \\ (F-f+1)f_{ij} + \dfrac{E\alpha r}{32\pi(1-\nu)\kappa\Delta\tau}\Bigg[(4r_{,l}n_l r_{,i}r_{,j} - r_{,j}n_i - r_{,i}n_j - \delta_{ij}r_{,l}n_l) \\ \times\left\{\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right) - \left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right)\right\} \\ -\left(r_{,j}n_i + r_{,i}n_j + \delta_{ij}\dfrac{(1+2\nu)}{(1-2\nu)}r_{,l}n_l\right)(E_1(s_f) - E_1(s_{f-1}))\Bigg] \end{bmatrix} \quad \text{(B.15)}$$

and

$$\int_{\tau_{f-1}}^{\tau_f} G_{ij}M^b\,\mathrm{d}\tau = \begin{bmatrix} -(F-f)g_{ij} - \dfrac{E\alpha r^2}{32\pi(1-\nu)\kappa\lambda\Delta\tau}\Bigg[\left(r_{,i}r_{,j} - \dfrac{\delta_{ij}}{2}\right)\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right) \\ -\left(r_{,i}r_{,j} + \dfrac{(1+2\nu)}{2(1-2\nu)}\delta_{ij}\right)\left\{\left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right) - (E_1(s_f) - E_1(s_{f-1}))\right\}\Bigg] \\ (F-f+1)g_{ij} + \dfrac{E\alpha r^2}{32\pi(1-\nu)\kappa\lambda\Delta\tau}\Bigg[\left(r_{,i}r_{,j} - \dfrac{\delta_{ij}}{2}\right)\left(\dfrac{(1-e^{-s_f})}{s_f^2} - \dfrac{(1-e^{-s_{f-1}})}{s_{f-1}^2}\right) \\ -\left(r_{,i}r_{,j} + \dfrac{(1+2\nu)}{2(1-2\nu)}\delta_{ij}\right)\left\{\left(\dfrac{e^{-s_f}}{s_f} - \dfrac{e^{-s_{f-1}}}{s_{f-1}}\right) - (E_1(s_f) - E_1(s_{f-1}))\right\}\Bigg] \end{bmatrix} \quad \text{(B.16)}$$

APPENDIX C: BOUNDED TERMS DUE TO SINGULAR INTEGRALS

Temperature derivative equation

The temperature derivative equation for a boundary point can be obtained by taking the X' point to x' in eqn (12). But, before taking the internal spatial point to the boundary, time integration should be done. The time integration for the derivatives of Θ and Q in eqn (12) is between τ_0 and τ_F. The time range τ_o–τ_F is divided into F equal time steps and the integration over the last time step can be written as

$$\int_\Gamma\left\{\int_{\tau_{(F-1)}}^{\tau_F}\frac{\partial Q}{\partial X'_i}\theta(x,\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma \quad \text{and} \quad \int_\Gamma\left\{\int_{\tau_{(F-1)}}^{\tau_F}\frac{\partial\Theta}{\partial X'_i}q(x,\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\Gamma.$$

By using constant interpolation over time, and noting that as $\tau \to \tau_F$, then

$$s \to \infty \quad \text{and} \quad e^{-s}, \ \frac{e^{-s}}{(\tau_F - \tau)} \to 0,$$

the two integrals can be written as follows:

$$\int_\Gamma\left\{\frac{r_{,k}n_k r_{,i}}{4\pi\kappa\Delta\tau}e^{-s_{F-1}} - \left(\frac{n_i}{2} - r_{,k}n_k r_{,i}\right)\frac{e^{-s_{F-1}}}{\pi r^2}\right\}\theta(x,\tau_F)\,\mathrm{d}\Gamma = \int_\Gamma A_i^F\theta(x,\tau_F)\,\mathrm{d}\Gamma \quad \text{(C.1)}$$

and

$$\int_\Gamma \left\{\frac{r_{,i}}{2\pi\lambda r} e^{-x_{F-1}}\right\} q(x,\tau_F)\,\mathrm{d}\Gamma = \int_\Gamma B_i^F q(x,\tau_F)\,\mathrm{d}\Gamma, \tag{C.2}$$

where $r = |X'-x|$.

At any smooth boundary point x', a semicircular Γ_ε^* of radius ε can be constructed for a boundary segment Γ_ε as shown in Fig. C.1. When the internal point moves to the boundary point the new boundary for x' is $(\Gamma-\Gamma_\varepsilon)+\Gamma_\varepsilon^*$. On a smooth boundary φ varies between 0 and π. Continuity requirements of hypersingular and strongly singular integrals can be seen from the paper by Krishnasamy *et al.* (1992). The order of singularity of the equation (C.1) is of $O(1/r^2)$ and of the eqn (C.2) is of $O(1/r)$. The q is assumed to be $C^{0,\alpha}$ continuous and θ to be $C^{1,\alpha}$ continuous. To regularise the integrals, the first term of the Taylor expansion for q and first two terms of the Taylor expansion for θ are subtracted.

The eqn (C.1) can be written as follows:

$$\begin{aligned}\lim_{X'\to x'}\int_\Gamma A_i^F\theta(x,\tau_F)\,\mathrm{d}\Gamma &= \lim_{\varepsilon\to 0}\int_{(\Gamma-\Gamma_\varepsilon)+\Gamma_\varepsilon^*} A_i^F\theta(x,\tau_F)\,\mathrm{d}\Gamma \\ &= \lim_{\varepsilon\to 0}\int_{\Gamma-\Gamma_\varepsilon} A_i^F\theta(x,\tau_F)\,\mathrm{d}\Gamma + \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^*} A_i^F\{\theta(x,\tau_F)-\theta(x',\tau_F)-\theta_{,i}(x',\tau_F)(x_i-x_i')\}\,\mathrm{d}\Gamma \\ &\quad + \theta(x',\tau_F)\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^*} A_i^F\,\mathrm{d}\Gamma + \theta_{,i}(x',\tau_F)\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^*} A_i^F(x_i-x_i')\,\mathrm{d}\Gamma. \end{aligned} \tag{C.3}$$

In eqn (C.3) the integral of (C.1) is divided into four parts. Of the four integrals, the first integral and the third integral together form the Hadamard principal value integral, the second integral goes to zero as $\varepsilon\to 0$, and the fourth integral gives a bounded term. The final integral is written as

$$\lim_{X'\to x'}\int_\Gamma A_i^F\theta(x,\tau_F)\,\mathrm{d}\Gamma = \lim_{\varepsilon\to 0} \oint\!\!\!\!=_\Gamma A_i^F\theta(x,\tau_F)\,\mathrm{d}\Gamma + \theta_{,i}(x',\tau_F)\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^*} A_i^F(x_i-x_i')\,\mathrm{d}\Gamma. \tag{C.4}$$

Similarly eqn (C.2) is as follows:

$$\begin{aligned}\lim_{X'\to x'}\int_\Gamma B_i^F q(x,\tau_F)\,\mathrm{d}\Gamma &= \lim_{\varepsilon\to 0}\int_{(\Gamma-\Gamma_\varepsilon)+\Gamma_\varepsilon^*} B_i^F q(x,\tau_F)\,\mathrm{d}\Gamma = \lim_{\varepsilon\to 0}\int_{\Gamma-\Gamma_\varepsilon} B_i^F q(x,\tau_F)\,\mathrm{d}\Gamma \\ &\quad + \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^*} B_i^F\{q(x,\tau_F)-q(x',\tau_F)\}\,\mathrm{d}\Gamma - \lambda\theta_{,i}(x',\tau_F)\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^*} B_i^F n_i(x)\,\mathrm{d}\Gamma; \end{aligned} \tag{C.5}$$

where the relation $q = -\lambda\theta_{,i}n_i$ is used and n_i is left inside the integral as it varies along the circular path. Of the three terms on the right-hand side of the eqn (C.5), the first term is a Cauchy principal value integral and the second term goes to zero as $\varepsilon\to 0$. The third integral gives a bounded term similar to the one in eqn (C.4). Then eqn (C.5) can be written as follows:

$$\lim_{X'\to x'}\int_\Gamma B_i^F q(x,\tau_F)\,\mathrm{d}\Gamma = \lim_{\varepsilon\to 0} \,{-}\!\!\!\!\!\int_\Gamma B_i^F q(x,\tau_F)\,\mathrm{d}\Gamma - \lambda\theta_{,i}(x',\tau_F)\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^*} B_i^F n_i(x)\,\mathrm{d}\Gamma. \tag{C.6}$$

To integrate the two bounded terms along Γ_ε^*, the following relations are necessary. They can be seen from Fig. C.1 to be:

$$\begin{aligned} r_{,1} = n_1 = \cos\varphi \quad &\text{and} \quad r_1 = (x_1-x_1') = \varepsilon\cos\varphi \\ r_{,2} = n_2 = \sin\varphi \quad &\text{and} \quad r_2 = (x_2-x_2') = \varepsilon\sin\varphi \\ r_{,i}n_i = 1 \quad &\text{and}\; \mathrm{d}\Gamma = \varepsilon\,\mathrm{d}\varphi. \end{aligned} \tag{C.7}$$

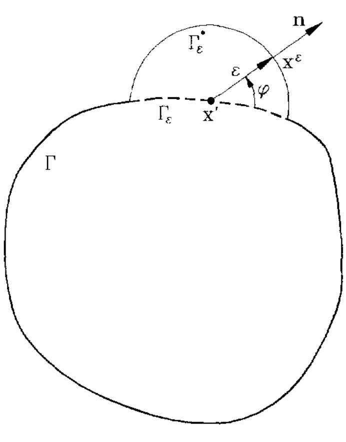

Fig. C.1. Source point on the boundary with semicircular extension.

After substituting the above relationships in eqns (C.4 and C.6) the following equalities are obtained

$$\lim_{x' \to x} \int_{\Gamma} A_i^F \theta(x, \tau_F) \, d\Gamma = \lim_{\varepsilon \to 0} \fint_{\Gamma} A_i^F \theta(x, \tau_F) \, d\Gamma + \frac{\theta_{,i}(x', \tau_F)}{4} \tag{C.8}$$

and

$$\lim_{x' \to x} \int_{\Gamma} B_i^F q(x, \tau_F) \, d\Gamma = \lim_{\varepsilon \to 0} \fint_{\Gamma} B_i^F q(x, \tau_F) \, d\Gamma - \frac{\theta_{,i}(x', \tau_F)}{4}. \tag{C.9}$$

Stress equation

Similarly to the temperature derivative equation, the stress equation also contains integrands of $O(1/r^2)$ and $O(1/r)$. By modifying the integration path around the boundary point x' to Γ_ε^* as shown in Fig. (C.1), the following can be obtained:

$$\lim_{x' \to x} \left\{ \int_{\Gamma} T_{kij} u_k(x, \tau_F) \, d\Gamma - \int_{\Gamma} U_{kij} t_k(x, \tau_F) \, d\Gamma \right\} = \fint_{\Gamma} T_{kij} u_k(x, \tau_F) \, d\Gamma - \fint_{\Gamma} U_{kij} t_k(x, \tau_F) \, d\Gamma$$
$$+ u_{k,l}(x', \tau_F) \int_{\Gamma_\varepsilon^*} T_{kij}(x_l - x_l') \, d\Gamma - \sigma_{kl}(x', \tau_F) \int_{\Gamma_\varepsilon^*} U_{kij} n_l \, d\Gamma \tag{C.10}$$

where the relation $t_k = \sigma_{kl} n_l$ is used. It can be proven from eqns (C.7) that

$$u_{k,l}(x', \tau_F) \int_{\Gamma_\varepsilon^*} T_{kij}(x_l - x_l') \, d\Gamma = -\frac{\mu}{8(1-v)} (2\varepsilon_{ij}(x', \tau_F) + \varepsilon_{kk}(x', \tau_F)\delta_{ij}) \tag{C.11}$$

and

$$\sigma_{kl}(x', \tau_F) \int_{\Gamma_\varepsilon^*} U_{kij} n_l \, d\Gamma = \frac{1}{8(1-v)} \left((3-4v)\sigma_{ij}(x', \tau_F) - \frac{(1-4v)}{2} \sigma_{kk}(x', \tau_F)\delta_{ij} \right). \tag{C.12}$$

From the eqns (C.10, C.11 and C.12) the following can be shown

$$u_{k,l}(x', \tau_F) \int_{\Gamma_\varepsilon^*} T_{kij}(x_l - x_l') \, d\Gamma - \sigma_{kl}(x', \tau_F) \int_{\Gamma_\varepsilon^*} U_{kij} n_l \, d\Gamma =$$
$$-\frac{\sigma_{ij}(x', \tau_F)}{2} - \frac{\mu(1+v)\alpha}{(1-2v)} \theta(x', \tau_F)\delta_{ij} + \frac{\mu(1+v)\alpha}{2(1-v)(1-2v)} \theta(x', \tau_F)\delta_{ij}. \tag{C.13}$$